1 What is a system?

Now that we have studied linear equations and linear inequalities, it is time to consider the question, “What if we solved two at once?” This probably sounds more difficult than it actually is. Let us briefly recall what a solution to an equation in one variable looks like. Suppose we have $4 - 2x = 3$. No problem here, this was one of the first things we learned in algebra. Start by subtracting 4 from both sides to get $-2x = -1$. Then divide through by $-2$ to get $x = \frac{1}{2}$. In other words, we had one equation with one variable, and solved for it. It was an equation (not an inequality), so the solution was a specific number, not a range of numbers. We could easily graph this solution on a number line; it would be a single, solid dot.

Now consider an equation in two variables, such as $y = -x + 5$. We know this to be a linear equation—that is, it is a line when graphed. We can also readily recognize this to be slope-intercept form, and thus $m = -1$ and $b = 5$. We could easily graph this line on a Cartesian plane by plotting the $y$-intercept and then counting out the slope and plotting a second point. The line that results contains an infinite number of points and we have just found two of them. Therefore, you can’t say that one particular point is the “solution” to that line, because it contains an infinite number of equally important points.

A system is more than one equation or inequality, considered together. One way to think of this is by plotting both of them on the same set of axes. We will start by looking at equations, and then move into inequalities later. The solution is the point(s) or region(s) where they intersect or overlap.
2 Systems of Linear Equations

We begin by considering $y = -x + 1$. Clearly, this is a line with slope $-1$ and $y$-intercept 1. It would look like this:

Now consider plotting a second line on the same coordinate plane. We will use $y = \frac{1}{2}x - 2$ for the sake of the example. Now our plot looks like this:

The first line remains unchanged, and the second line is exactly what we thought it would be with a slope of $\frac{1}{2}$ and a $y$-intercept of $-2$. The interesting thing is that they intersect, at one single point specifically. That point is the **solution** to this system of equations. As we will see below, there are different types of solutions to systems of linear equations; a single point is one of those types. We will now learn how to find that intersection point.
2.1 Solving by Substitution

We will continue using the same equations we just graphed.

\[ y = -x + 1 \]
\[ y = \frac{1}{2}x - 2 \]

If you think about it, the intersection point we seek is the only place where both lines have the same \( (x, y) \) value; that is, we are looking for the \( x \) and \( y \) values that satisfy both equations. Both equations are in slope-intercept form, so they are both equal to \( y \) and we can use the transitive property to set them equal to each other.

\[-x + 1 = \frac{1}{2}x - 2\]

You should now notice that we have eliminated \( y \) from the equation entirely, leaving us with one equation and one unknown, \( x \). Go through the typical steps to solve this:

\[-x + 1 = \frac{1}{2}x - 2\]
\[-x = \frac{1}{2}x - 2 - 1\]
\[-x = \frac{1}{2}x - 3\]
\[-x + \frac{1}{2}x = -3\]
\[-\frac{3}{2}x = -3\]
\[x = -3(-\frac{2}{3})\]
\[x = 2\]

Looking back at the graph in the previous section, we notice that the \( x \)-coordinate of the lines’ intersection point is 2. Since this \( x \) value satisfies both equations at the same point, we can find the \( y \)-coordinate by choosing either of the starting equations, plugging in \( x = 2 \), and solving for \( y \). I typically choose the one with easier math. In this case, that is first one, which gives \( y = -2 + 1 = -1 \) and we have found the \( y \)-coordinate of the intersection point. Notice that indeed the two lines cross at \((2, -1)\).

Now let us consider a second example which isn’t as easy. The previous example consisted of a system in which both lines were already given in slope-intercept form, making it quite easy to equate them. If one or more of the
equations is not given in slope-intercept form, you simply pick one to solve for a variable, and then continue as discussed above. Note that the choice of variable and equation to solve first is arbitrary.

Consider the system $-5x - 8y = 17$ and $2x - 7y = -17$. We quickly recognize that these lines are both given in standard form. You can arbitrarily choose either equation and solve it for either variable; your result will be the same. You will still plug it into the other equation regardless of the starting point you choose. Let’s choose the second equation and solve for $x$.

\[
2x - 7y = -17
\]
\[
2x = 7y - 17
\]
\[
x = \frac{7}{2}y - \frac{17}{2}
\]

Now we can plug this in for $x$ in the first equation and solve for $y$.

\[
-5\left(\frac{7}{2}y - \frac{17}{2}\right) - 8y = 17
\]
\[
-\frac{35}{2}y + \frac{85}{2} - 8y = 17
\]
\[
-\frac{51}{2}y = -\frac{51}{2}
\]
\[
y = 1
\]

Finally, this can be plugged into either equation to find the $x$-coordinate of the solution. For this example, we will arbitrarily choose the first equation.

\[
-5x - 8(1) = 17
\]
\[
-5x = 25
\]
\[
x = -5
\]

Thus, our solution is $(-5, 1)$ and we have found the intersection point of the two lines.

Aside: I have a recommendation that will serve you well in future math classes when analysis is a bigger focus than algebra skills. If you only care about one of the variables, you should solve first for the variable you don’t care about, thus getting rid of it when substituting into the other equation. That leaves you with only the variable of interest. This is just a quick trick to help you out when large calculations are looming. It will keep you from doing a lot of unnecessary work.
2.2 Solving by Elimination

Another method of solving a system of equations is by eliminating one of the variables by adding or subtracting the two equations in their entirety. This is especially useful when the equations are complicated, or when one of the variables has matching coefficients in the two equations. You can also multiply or divide an equation by a number to create a matching coefficient, which we will do in the next section. Consider the system 

\[-4x - 2y = -12\]
\[4x + 8y = -24\]

Notice that the \(x\)-coefficients are equal but opposite. If you disregard everything else momentarily, you can see that \(-4 + 4 = 0\). This is the same idea we are going to use to “eliminate” the \(x\) terms. Line up the corresponding terms of each equation like a traditional addition problem—the \(x\)-terms, \(y\)-terms, and constants.

\[
\begin{align*}
-4x - 2y &= -12 \\
4x + 8y &= -24
\end{align*}
\]

Now you simply add or subtract each term from its corresponding mate. You choose addition or subtraction so that the matching coefficients will disappear. For example, we need to add here, so that we have \(-4 + 4 = 0\). If both coefficients had been \(-4\), then we would need to subtract, thereby getting \(-4 - 4 = -4 + 4 = 0\). If this seems confusing, just choose some numbers and play around with it until it makes sense. When we add all corresponding terms, we get

\[
0x + 6y = -36
\]

or in other words \(6y = -36\) and we see that the \(x\)-terms have gone away. Simplifying is easy to do, and we get \(y = -6\). Notice that we now have the \(y\)-coordinate of the solution, which can now be plugged into either equation to find the \(x\)-coordinate of the solution. We arbitrarily choose the second equation, which becomes

\[
\begin{align*}
4x + 8(-6) &= -24 \\
4x - 48 &= -24 \\
4x &= 24 \\
x &= 6
\end{align*}
\]

and we have found the solution to be \((6, -6)\).
2.2.1 Multiplying to Create Matching Coefficients

Alternatively, you may be able to use elimination by first multiplying or dividing through an entire equation to create “matching” coefficients. This is beneficial when you have corresponding coefficients that are factors of each other. An example would be the system $5x + 4y = −14$ and $3x + 6y = 6$. We notice that there is a relationship between the $y$-coefficients: 4 and 6 are related to each other by a factor of $\frac{3}{2}$. Therefore, we can exploit this to create matching coefficients that can then be eliminated. Multiplying the first equation through by $\frac{3}{2}$ gives

\[
\left(\frac{3}{2}\right)(5x + 4y) = (-14)\left(\frac{3}{2}\right)
\]

\[
\frac{15}{2}x + \frac{12}{2}y = -\frac{42}{2}
\]

\[
\frac{15}{2}x + 6y = -21
\]

Now are in the position to add the two equations together because we have created matching corresponding coefficients on the $y$ terms. Since they are both positive, we will have to subtract the second equation, since if we added them, we would get $12y$ instead of $0y$.

\[
\frac{15}{2}x + 6y = -21
\]

\[
3x + 6y = 6
\]

Subtracting each of the corresponding terms gives

\[
\frac{9}{2}x + 0y = -27
\]

or more simply

\[
\frac{9}{2}x = -27
\]

and dividing through by $\frac{9}{2}$ yields

\[
x = -6
\]

Now we can plug back into either equation to solve for the remaining $y$-coordinate, and business is back to normal.
2.3 Types of Solutions

There are three types of solutions to a system of two linear equations, which is the scope of our studies for the time being. Having already discussed the first type, a single point, let us now turn our attention to the other two types of solutions: infinite or none. Consider for a moment what these might look like. If a single solution is a single point of intersection between two lines, the infinite solutions would be where the lines “overlap” (i.e., they’re the same line) and no solution would be where the two lines are parallel and never intersect. If you get a nonsensical solution like $4 = 9$ then your system has no solution, and the lines never intersect (i.e., they are parallel). If your math boils down to a true statement like $6 = 6$ then you have infinite solutions, and the lines are on top of each other (i.e., the same line).

3 Graphing Systems of Linear Inequalities

Inequalities receive a lightly different treatment, although the logic is the same. As we have seen with two lines, this can be a single point, infinite solutions, or no solution. Now we will consider what this would look like with two inequalities. This will result in overlapping regions instead of points. There are three steps to solving a system of inequalities:

1. Get both inequalities into slope-intercept form.

2. Plot the boundary. (solid or dashed)

3. Determine which side to shade. (check a point)

Let’s consider the system $y \geq \frac{2}{3}x + 3$ and $y > -\frac{4}{3}x - 3$. It’s easy to see these are both already in slope-intercept form. Pretend they are regular equations in $y = mx + b$ form and graph the first one. Since it is “greater than or equal to” it is a solid line when plotted. Finally check a point to see which side to shade. The origin is always an easy one to calculate. Plugging in $(0, 0)$ gives $0 \geq 0 + 3$ which is clearly false, so therefore you shade the side opposite to the check point. If the check point had given you a true statement, then you would shade the side that includes the check point. Repeat the process for the second inequality. Notice that it is “greater than” (not or equal to) so it becomes a dashed line when plotted. That tells you that the boundary itself isn’t part of the solution. Here is what the final product would look like:
Notice that one of the boundaries is dashed, and one is solid. The solution to the system is the region where the two shadings overlap, including the solid boundary and not including the dashed boundary.